A first-order differential double subordination with applications[☆]

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ABSTRACT

Let q_1 and q_2 belong to a certain class of normalized analytic univalent functions in the open unit disk of the complex plane. Sufficient conditions are obtained for normalized analytic functions p to satisfy the double subordination chain $q_1(z) \prec p(z) \prec q_2(z)$. The differential sandwich-type result obtained is applied to normalized univalent functions and to Φ-like functions.

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1. Introduction

Let $\mathscr{H}$ be the class consisting of analytic functions in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ of the complex plane $\mathbb{C}$. For $a \in \mathbb{C}$, let $\mathscr{H}[a, n] := \{f \in \mathscr{H} : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots\}$, and $\mathscr{A} := \{f \in \mathscr{H} : f(0) = 0, f'(0) = 1\}$. A function $f \in \mathscr{H}$ is said to be subordinate to an analytic function $g \in \mathscr{H}$, or g superordinates f, written as $f(z) \prec g(z) (z \in \mathbb{D})$, if there exists a Schwarz function w, analytic in $\mathbb{D}$ with $w(0) = 0$ and $|w(z)| < 1$, satisfying $f(z) = g(w(z))$. If the function g is univalent in $\mathbb{D}$, then $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$. An exposition on the widely used theory of differential subordination, developed in the main by Miller and Mocanu, with numerous applications to univalent functions can be found in their monograph [1]. Miller and Mocanu [2] also introduced the dual concept of differential superordination. Let $p, h \in \mathscr{H}$ and $\phi(r, s, t; z) : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$. If p and $\phi(p(z), zp'(z), z^2 p''(z); z)$ are univalent and p satisfies the second-order superordination

$$h(z) \prec \phi(p(z), zp'(z), z^2 p''(z); z), \tag{1}$$

then p is a solution of the differential superordination (1). An analytic function q is called a *subordinant* if $q \prec p$ for all p satisfying (1). A univalent subordinant $\widetilde{q}$ satisfying $q \prec \widetilde{q}$ for all subordinants q of (1) is said to be the best subordinant. Miller and Mocanu [2] obtained conditions on h, q and ϕ for which the following differential implication holds:

$$h(z) \prec \phi(p(z), zp'(z), z^2 p''(z); z) \Rightarrow q(z) \prec p(z).$$

Using these results, Bulboacă gave a treatment on certain classes of first-order differential superordinations [3,4], as well as superordination-preserving integral operators [5]. Ali et al. [6] gave several applications of first-order differential subordination and superordination to obtain sufficient conditions for normalized analytic functions f to satisfy $q_1(z) \prec zf'(z)/f(z) \prec q_2(z)$, where q_1 and q_2 are given univalent analytic functions in $\mathbb{D}$. In [7], they have also applied differential superordination to functions defined by means of linear operators. Recently Ali and Ravichandran [8] investigated first-order superordination to a class of meromorphic α-convex functions. Several differential subordination and superordination associated with various linear operators were also investigated in [9].

Generalizing the familiar starlike and convex functions, Lewandowski et al. [10] introduced γ-starlike functions consisting of $f \in \mathscr{A}$ satisfying the inequality

$$\mathrm{Re}\left(\left(\frac{zf'(z)}{f(z)}\right)^{1-\gamma}\left(1+\frac{zf''(z)}{f'(z)}\right)^{\gamma}\right) > 0.$$

These functions are starlike. With $p(z) := zf'(z)/f(z)$, to show that γ-starlike functions are indeed starlike, is to analytically make the implication

$$\mathrm{Re}\left(p(z)\left(1+\frac{zp'(z)}{p^2(z)}\right)^{\gamma}\right) > 0 \Rightarrow \mathrm{Re}\,p(z) > 0.$$

Following the work of Lewandowski et al. [10,11], Kanas et al. [12] determined conditions on p and h satisfying

$$p(z)\left(1+\frac{zp'(z)}{p(z)}\right)^{\alpha} \prec h(z) \Rightarrow p(z) \prec h(z)$$

for a fixed $\alpha \in [0, 1]$. Lecko [13] (see [12] for a symmetric version) investigated the more general subordination

$$p(z)\left(1+\frac{zp'(z)}{p(z)}\varphi(p(z))\right)^{\alpha} \prec h(z) \Rightarrow p(z) \prec h(z).$$

Singh and Gupta [14] subsequently investigated the following first-order differential subordination that included the important Briot–Bouquet differential subordination.

$$(p(z))^{\alpha}\left(p(z)+\frac{zp'(z)}{\beta p(z)+\gamma}\right)^{\mu} \prec (q(z))^{\alpha}\left(q(z)+\frac{zq'(z)}{\beta q(z)+\gamma}\right)^{\mu} \Rightarrow p(z) \prec q(z).$$

For a closely related class, see [15].

The present paper investigates differential subordination and superordination implications of expressions similar to the form considered above by Singh and Gupta [14]. Special cases of the results obtained include one involving the expression $\alpha p^2(z)+(1-\alpha)p(z)+\alpha zp'(z)$, a result which cannot be deduced from the work of Singh and Gupta [14]. The sandwich-type results obtained in our present investigation are then applied to normalized analytic univalent functions and to Φ-like functions.

The following definition and results will be required.

Lemma 1.1 (*cf. Miller and Mocanu [1, Theorem 3. 4h, p. 132]*)**.** *Let q be univalent in the unit disk $\mathbb{D}$, and let ϑ and φ be analytic in a domain $D \supset q(\mathbb{D})$ with $\varphi(w) \neq 0$, $w \in q(\mathbb{D})$. With $Q(z) := zq'(z)\varphi(q(z))$, let $h(z) := \vartheta(q(z))+Q(z)$. Suppose that Q is starlike univalent in $\mathbb{D}$ and*

$$\mathrm{Re}\left(\frac{zh'(z)}{Q(z)}\right) > 0 \quad (z \in \mathbb{D}).$$

If p is analytic in $\mathbb{D}$ with $p(0) = q(0)$, $p(\mathbb{D}) \subset D$ and

$$\vartheta(p(z))+zp'(z)\varphi(p(z)) \prec \vartheta(q(z))+zq'(z)\varphi(q(z)),$$

then $p(z) \prec q(z)$, and q is the best dominant.

Definition 1.2 (*[2, Definition 2, p. 817]*)**.** *Denote by $\mathscr{Q}$ the set of all functions f that are analytic and injective on $\overline{\mathbb{D}} - E(f)$, where*

$$E(f) = \left\{\zeta \in \partial\mathbb{D} : \lim_{z\to\zeta} f(z) = \infty\right\},$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{D} - E(f)$.

Lemma 1.3 (*[4]*)**.** *Let q be univalent in the unit disk $\mathbb{D}$, ϑ and φ be analytic in a domain D containing $q(\mathbb{D})$. Suppose that $\mathrm{Re}[\vartheta'(q(z))/\varphi(q(z))] > 0$ for $z \in \mathbb{D}$ and $zq'(z)\varphi(q(z))$ is starlike univalent in $\mathbb{D}$. If $p \in \mathscr{H}[q(0), 1] \cap \mathscr{Q}$ with $p(\mathbb{D}) \subseteq D$, and $\vartheta(p(z))+zp'(z)\varphi(p(z))$ is univalent in $\mathbb{D}$, then*

$$\vartheta(q(z))+zq'(z)\varphi(q(z)) \prec \vartheta(p(z))+zp'(z)\varphi(p(z))$$

implies $q(z) \prec p(z)$, and q is the best subordinant.

2. A sandwich theorem

Our main result involves the following class of functions.

Definition 2.1. Let α and μ be fixed numbers with $0 < \mu \leq 1, \alpha + \mu \geq 0$. Also let β, γ and δ be complex numbers with $\beta \neq 0$. The class $\mathscr{R}(\alpha, \beta, \gamma, \delta, \mu)$ consists of analytic functions p with $p(0) = 1, p(z) \neq 0$ in $\mathbb{D}$, and are such that the functions

$$P(z) := (p(z))^{\alpha} \left(p(z) + \delta + \frac{zp'(z)}{\beta p(z) + \gamma} \right)^{\mu} \quad (z \in \mathbb{D})$$

are well-defined in $\mathbb{D}$. (Here the powers are principal values.)

By making use of Lemma 1.1, the following result is derived.

Theorem 2.2. Let $q \in \mathscr{R}(\alpha, \beta, \gamma, \delta, \mu)$ be analytic and univalent in $\mathbb{D}$. Set

$$R(z) := \frac{zq'(z)}{\beta q(z) + \gamma} \quad (z \in \mathbb{D}). \tag{2}$$

Assume that

$$\operatorname{Re} \left((\beta q(z) + \gamma) \left(1 + \frac{\alpha}{\mu} + \frac{\alpha \delta}{\mu q(z)} \right) \right) > 0 \quad (z \in \mathbb{D}), \tag{3}$$

and

$$\operatorname{Re} \left(\frac{\alpha}{\mu} \frac{zq'(z)}{q(z)} + \frac{zR'(z)}{R(z)} \right) > 0 \quad (z \in \mathbb{D}). \tag{4}$$

If $p \in \mathscr{R}(\alpha, \beta, \gamma, \delta, \mu)$ satisfies

$$(p(z))^{\alpha} \left(p(z) + \delta + \frac{zp'(z)}{\beta p(z) + \gamma} \right)^{\mu} \prec (q(z))^{\alpha} \left(q(z) + \delta + \frac{zq'(z)}{\beta q(z) + \gamma} \right)^{\mu}, \tag{5}$$

then $p(z) \prec q(z)$, and q is the best dominant.

Proof. We first write the differential subordination (5) as

$$(p(z))^{\frac{\alpha}{\mu}+1} + \delta (p(z))^{\frac{\alpha}{\mu}} + (p(z))^{\frac{\alpha}{\mu}} \frac{zp'(z)}{\beta p(z) + \gamma} \prec (q(z))^{\frac{\alpha}{\mu}+1} + \delta (q(z))^{\frac{\alpha}{\mu}} + (q(z))^{\frac{\alpha}{\mu}} \frac{zq'(z)}{\beta q(z) + \gamma}.$$

Define the functions ϑ and φ by

$$\vartheta(w) := w^{\frac{\alpha}{\mu}+1} + \delta w^{\frac{\alpha}{\mu}} \quad \text{and} \quad \varphi(w) := \frac{w^{\frac{\alpha}{\mu}}}{\beta w + \gamma}.$$

Since $q \in \mathscr{R}(\alpha, \beta, \gamma, \delta, \mu)$, then $q(z) \neq 0$ and therefore $\varphi(w) \neq 0$ when $w \in q(\mathbb{D})$. Also φ and ϑ are analytic in a domain containing $q(\mathbb{D})$. Define the function

$$Q(z) := zq'(z) \varphi(q(z)) = (q(z))^{\frac{\alpha}{\mu}} \frac{zq'(z)}{\beta q(z) + \gamma} = (q(z))^{\frac{\alpha}{\mu}} R(z),$$

where R is given by (2). It follows from (4) that

$$\operatorname{Re} \frac{zQ'(z)}{Q(z)} = \Re \left(\frac{\alpha}{\mu} \frac{zq'(z)}{q(z)} + \frac{zR'(z)}{R(z)} \right) > 0,$$

and so Q is a starlike function. Now define h by

$$h(z) := \vartheta(q(z)) + Q(z) = (q(z))^{\frac{\alpha}{\mu}+1} + \delta (q(z))^{\frac{\alpha}{\mu}} + Q(z).$$

In view of the assumptions (3) and (4), it follows that

$$\operatorname{Re} \frac{zh'(z)}{Q(z)} = \operatorname{Re} \left\{ (\beta q(z) + \gamma) \left(1 + \frac{\alpha}{\mu} + \frac{\alpha \delta}{\mu q(z)} \right) + \frac{\alpha}{\mu} \frac{zq'(z)}{q(z)} + \frac{zR'(z)}{R(z)} \right\} > 0 \quad (z \in \mathbb{D}).$$

The result is now deduced from Lemma 1.1. $\quad\square$

Example 2.3. Let $q : \mathbb{D} \to \mathbb{C}$ be defined by $q(z) = (1 + Az)/(1 + Bz)$ with $-1 < B < A \leq 1$. It is evident that $q \in \mathscr{R}(\alpha, \beta, \gamma, \delta, \mu)$ whenever

$$\delta + \frac{1 - A}{1 - B} > \frac{A - B}{(1 - B)| \, |\beta + \gamma| - |\beta A + \gamma B| \, |}.$$

With additional constraints on the parameters, there exist functions q satisfying the hypothesis of Theorem 2.2. For instance, in addition to the above condition, assuming that all the parameters $\alpha, \beta, \gamma, \delta$, and μ are positive with

$$\frac{1 - 2A}{1 - A} > \frac{|\beta A + \gamma B|}{|\beta + \gamma - |\beta A + \gamma B||},$$

then q satisfies the conditions of Theorem 2.2.

By a similar application of Lemma 1.3, the following result can be established, which we state without proof.

Theorem 2.4. *Let $q \in \mathscr{R}(\alpha, \beta, \gamma, \delta, \mu)$ be as in Theorem 2.2. Let $p \in \mathscr{R}(\alpha, \beta, \gamma, \delta, \mu)$ satisfies $p \in \mathscr{H} \cap \mathscr{Q}$ and $(p(z))^{\frac{\alpha}{\mu} + 1} + \delta(p(z))^{\frac{\alpha}{\mu}} + (p(z))^{\frac{\alpha}{\mu}} \frac{zp'(z)}{\beta p(z) + \gamma}$ be univalent. If p satisfies*

$$(q(z))^{\alpha} \left(q(z) + \delta + \frac{zq'(z)}{\beta q(z) + \gamma} \right)^{\mu} \prec (p(z))^{\alpha} \left(p(z) + \delta + \frac{zp'(z)}{\beta p(z) + \gamma} \right)^{\mu},$$

then $q(z) \prec p(z)$, and q is the best subordinant.

Combining Theorems 2.2 and 2.4, the following "sandwich theorem" is obtained.

Theorem 2.5. *Let $q_i \in \mathscr{R}(\alpha, \beta, \gamma, \delta, \mu)$ $(i = 1, 2)$ be analytic and univalent in $\mathbb{D}$. Set*

$$R_i(z) := \frac{zq_i'(z)}{\beta q_i(z) + \gamma} \quad (i = 1, 2; z \in \mathbb{D}),$$

$$h_i(z) := (q_i(z))^{\alpha} \left(q_i(z) + \delta + \frac{zq_i'(z)}{\beta q_i(z) + \gamma} \right)^{\mu} \quad (i = 1, 2).$$

Assume that

$$\mathrm{Re}\left((\beta q_i(z) + \gamma) \left(1 + \frac{\alpha}{\mu} + \frac{\alpha\delta}{\mu q_i(z)} \right) \right) > 0 \quad (z \in \mathbb{D})$$

and

$$\mathrm{Re}\left(\frac{\alpha}{\mu} \frac{zq_i'(z)}{q_i(z)} + \frac{zR_i'(z)}{R_i(z)} \right) > 0 \quad (i = 1, 2; z \in \mathbb{D}).$$

If $p \in \mathscr{R}(\alpha, \beta, \gamma, \delta, \mu)$ satisfies $p \in \mathscr{H} \cap \mathscr{Q}$ and $(p(z))^{\frac{\alpha}{\mu} + 1} + \delta(p(z))^{\frac{\alpha}{\mu}} + (p(z))^{\frac{\alpha}{\mu}} \frac{zp'(z)}{\beta p(z) + \gamma}$ is univalent, then

$$h_1(z) \prec (p(z))^{\alpha} \left(p(z) + \delta + \frac{zp'(z)}{\beta p(z) + \gamma} \right)^{\mu} \prec h_2(z) \tag{6}$$

implies $q_1(z) \prec p(z) \prec q_2(z)$. Further q_1 and q_2 are the best subordinant and the best dominant respectively.

3. Applications to univalent functions

By use of Theorem 2.5, the following result is obtained.

Theorem 3.1. *Let α, μ be fixed numbers with $0 < \mu \leq 1, \alpha + \mu > 0$, and $\lambda \in \mathbb{C}$. Let $f, g \in \mathscr{A}$, and $q_i(z) = zg_i'(z)/g_i(z)$ $(i = 1, 2)$ be univalent in $\mathbb{D}$ satisfying*

$$\mathrm{Re}\left(\frac{1}{\lambda} q_i(z) \right) > 0$$

and

$$\mathrm{Re}\left(\left(\frac{\alpha}{\mu} - 1 \right) \frac{zq_i'(z)}{q_i(z)} + 1 + \frac{zq_i''(z)}{q_i'(z)} \right) > 0.$$

Let

$$h_i(z) := \left(\frac{zg_i'(z)}{g_i(z)}\right)^\alpha \left((1-\lambda)\frac{zg_i'(z)}{g_i(z)} + \lambda\left(1 + \frac{zg_i''(z)}{g_i'(z)}\right)\right)^\mu \quad (i = 1, 2).$$

If $f \in \mathscr{A}$ satisfies $0 \neq \frac{zf'(z)}{f(z)} \in \mathscr{H}[1, 1] \cap \mathscr{Q}$ and $(\frac{zf'(z)}{f(z)})^\alpha((1-\lambda)\frac{zf'(z)}{f(z)} + \lambda(1 + \frac{zf''(z)}{f'(z)}))^\mu$ is univalent in $\mathbb{D}$, then

$$h_1(z) \prec \left(\frac{zf'(z)}{f(z)}\right)^\alpha \left((1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right)\right)^\mu \prec h_2(z)$$

implies

$$\frac{zg_1'(z)}{g_1(z)} \prec \frac{zf'(z)}{f(z)} \prec \frac{zg_2'(z)}{g_2(z)}.$$

Proof. The result follows from Theorem 2.5 by taking $\gamma = \delta = 0$, $\beta = 1/\lambda$, and

$$p(z) := \frac{zf'(z)}{f(z)} \quad \text{and} \quad q_i(z) := \frac{zg_i'(z)}{g_i(z)} \quad (i = 1, 2). \quad \square$$

The following two corollaries are immediate consequences of Theorem 2.5 (or Theorem 3.1).

Corollary 3.2 ([6])**.** *Let $\alpha \in \mathbb{C}$, and $q_i(z) \neq 0$ ($i = 1, 2$) be univalent in $\mathbb{D}$. Assume that $\mathrm{Re}[\overline{\alpha} q_i(z)] > 0$ for $i = 1, 2$ and $zq_i'(z)/q_i(z)$ ($i = 1, 2$) is starlike univalent in $\mathbb{D}$. If $f \in \mathscr{A}$, $0 \neq zf'(z)/f(z) \in \mathscr{H}[1, 1] \cap \mathscr{Q}$, $(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha(1 + \frac{zf''(z)}{f'(z)})$ is univalent in $\mathbb{D}$, then*

$$q_1(z) + \alpha\frac{zq_1'(z)}{q_1(z)} \prec (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec q_2(z) + \alpha\frac{zq_2'(z)}{q_2(z)}$$

implies

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z).$$

Further q_1 and q_2 are the best subordinant and best dominant respectively.

Corollary 3.3 ([6])**.** *Let $q_i(z) \neq 0$ be univalent in $\mathbb{D}$ with $\mathrm{Re}\, q_i(z) > 0$. Let $zq_i'(z)/q_i^2(z)$ be starlike univalent in $\mathbb{D}$ for $i = 1, 2$. If $f \in \mathscr{A}$, $0 \neq zf'(z)/f(z) \in \mathscr{H}[1, 1] \cap Q$, $\frac{1+zf''(z)/f'(z)}{zf'(z)/f(z)}$ is univalent in $\mathbb{D}$, then*

$$1 + \frac{zq_1'(z)}{q_1^2(z)} \prec \frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} \prec 1 + \frac{zq_2'(z)}{q_2^2(z)}$$

implies $q_1(z) \prec zf'(z)/f(z) \prec q_2(z)$. Further q_1 and q_2 are the best subordinant and best dominant respectively.

Another application of Theorem 2.5 yields the following result.

Corollary 3.4 ([6])**.** *Let q_1 and q_2 be convex univalent in $\mathbb{D}$. Let $0 \neq \alpha \in \mathbb{C}$, and assume that $\mathrm{Re}\, q_i(z) > \mathrm{Re}\frac{\alpha-1}{2\alpha}$ for $i = 1, 2$. If $f \in \mathscr{A}$, $zf'(z)/f(z) \in \mathscr{H}[1, 1] \cap \mathscr{Q}$, $\frac{zf'(z)}{f(z)} + \alpha\frac{z^2f''(z)}{f(z)}$ is univalent in $\mathbb{D}$, then*

$$(1-\alpha)q_1(z) + \alpha q_1^2(z) + \alpha zq_1'(z) \prec \frac{zf'(z)}{f(z)}\left(1 + \alpha\frac{zf''(z)}{f'(z)}\right) \prec (1-\alpha)q_2(z) + \alpha q_2^2(z) + \alpha zq_2'(z)$$

implies

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z).$$

Further q_1 and q_2 are the best subordinant and best dominant respectively.

4. Application to Φ-like functions

Let Φ be an analytic function in a domain containing $f(\mathbb{D})$, $\Phi(0) = 0$ and $\Phi'(0) > 0$. A function $f \in \mathscr{A}$ is called Φ-like if

$$\operatorname{Re} \frac{zf'(z)}{\Phi(f(z))} > 0 \quad (z \in \mathbb{D}).$$

This concept was introduced by Brickman [16] and it was shown that an analytic function $f \in \mathscr{A}$ is univalent if and only if f is Φ-like for some Φ. When $\Phi(w) = w$ and $\Phi(w) = \lambda w$, the Φ-like function f is respectively starlike and spiral-like of type $\arg \lambda$. Ruscheweyh [17] introduced and studied the following general class of Φ-like functions.

Definition 4.1. Let Φ be analytic in a domain containing $f(\mathbb{D})$, $\Phi(0) = 0$, $\Phi'(0) = 1$ and $\Phi(\omega) \neq 0$ for $\omega \in f(\mathbb{D}) - \{0\}$. Let q be a fixed analytic function in $\mathbb{D}$, with $q(0) = 1$. A function $f \in \mathscr{A}$ is called Φ-like with respect to q if

$$\frac{zf'(z)}{\Phi(f(z))} \prec q(z) \quad (z \in \mathbb{D}).$$

Theorem 4.2. *Let $\alpha \neq 0$ be a complex number and q_i ($i = 1, 2$) be convex univalent in $\mathbb{D}$. Define h_i by*

$$h_i(z) := \alpha q_i^2(z) + (1 - \alpha)q_i(z) + \alpha z q_i'(z) \quad (i = 1, 2),$$

and suppose that

$$\operatorname{Re}\left(\frac{1 - \alpha}{\alpha} + 2q_i(z) \right) > 0 \quad (i = 1, 2; z \in \mathbb{D}).$$

If $f \in \mathscr{A}$ satisfies $f \in \mathscr{H}[1, 1] \cap \mathscr{Q}$ and $\frac{zf'(z)}{\Phi(f(z))}(1 + \frac{\alpha zf''(z)}{f'(z)} + \frac{\alpha z(f'(z) - (\Phi(f(z)))')}{\Phi(f(z))})$ is univalent in $\mathbb{D}$, then

$$h_1(z) \prec \frac{zf'(z)}{\Phi(f(z))} \left(1 + \frac{\alpha zf''(z)}{f'(z)} + \frac{\alpha z(f'(z) - (\Phi(f(z)))')}{\Phi(f(z))} \right) \prec h_2(z) \tag{7}$$

implies

$$q_1(z) \prec \frac{zf'(z)}{\Phi(f(z))} \prec q_2(z).$$

Further q_1 and q_2 are the best subordinant and the best dominant respectively.

Proof. Define the function p by

$$p(z) := \frac{zf'(z)}{\Phi(f(z))} \quad (z \in \mathbb{D}). \tag{8}$$

Then the function p is analytic in $\mathbb{D}$ with $p(0) = 1$. From (8), it follows that

$$\frac{zf'(z)}{\Phi(f(z))} \left(1 + \frac{\alpha zf''(z)}{f'(z)} + \frac{\alpha z(f'(z) - (\Phi(f(z)))')}{\Phi(f(z))} \right) = p(z) \left(1 + \alpha \left(\frac{zp'(z)}{p(z)} - 1 \right) + \alpha p(z) \right)$$

$$= \alpha p^2(z) + (1 - \alpha)p(z) + \alpha z p'(z). \tag{9}$$

Substituting (9) in the subordination (7) yields

$$h_1(z) \prec \alpha p^2(z) + (1 - \alpha)p(z) + \alpha z p'(z) \prec h_2(z).$$

The result now follows from Theorem 2.5. $\square$

Remark 1. When $\Phi(w) = w$, Theorem 4.2 reduces to Corollary 3.4.

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